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# Interaction Analysis in MANOVA

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## Abstract

Simultaneous test procedures (STPS for short) in the context of the unrestricted full rank general linear multivariate model for population cell means are introduced and utilized to analyze interactions in factorial designs. By appropriate choice of an implying hypothesis, it is shown how to test overall main effects, interactions, simple main, simple simple main and simple interaction effects. Implicit in the STP procedure is the control of the type I error rate for an implied family of tests. A set of simultaneous confidence bounds around contrasts of population cell means can be constructed without increasing the type I error rate. Numerical examples are given to illustrate the method.

# Interaction Analysis in MANOVA

The purpose of this article is to present a technique (Gabriel, 1968, 1969a, 1969b) which can be used to examine significant interaction effects in multivariate analysis. In the case of one dependent variable, it is applicable to univariate designs. Kirk (1968) and Winer (1971) present computational formulas for carrying out tests of simple main effects after finding a significant interaction in two-way designs and simple main effects, simple simple main effects and simple interactions for three-way factorials depending on the pattern of interactions. The extension and formulation of hypotheses and, in some cases, a certain class of simultaneous confidence intervals for these effects is given in the context of the general multivariate population cell means model. In the case of one dependent variable, the confidence bounds reduce to those of Scheffé.

A simultaneous test procedure requires choosing an overall null hypothesis which will imply all hypotheses of interest. The overall hypothesis together with the implied hypotheses are thought of as a family of tests. All tests of hypotheses associated with the family are carried out with respect to the critical value associated with the overall null hypothesis. By choosing an $\alpha$-level critical value, the overall familywise type I error rate is no more than $\alpha$. Depending on the family of hypotheses being considered, it is possible to construct

simultaneous confidence intervals for a given subset of the family. The relationship between a 100 (1 - α)% confidence interval and the α-level test of significance is well known. Gabriel (1964) has shown that the STP in the univariate case includes the Scheffé intervals and allows the testing of other hypotheses not covered by the Scheffé technique. Analogously, in the multivariate case, a certain subset of hypotheses have simultaneous confidence intervals associated with them (Gabriel, 1968).

To indicate how one might use the STP, consider a factorial design. In the context of the full rank population cell means model take as the overall hypothesis the equality of the population cell means. This hypothesis implies the usual hypotheses associated with overall main effects, first order interactions, second order interactions, etc. The interpretation of main effects is dependent on the pattern of significant interactions. In a two-way MANOVA design, finding a significant interaction is a signal to explore simple row and column effects hypotheses. These hypotheses involve testing differences among population means of one factor for fixed levels of the other factor.

In a three-way MANOVA design, one can analyze a significant second order interaction by formulating hypotheses associated with simple interactions, i.e., first order interactions obtained by fixing levels of a factor. Having found which simple interaction hypotheses are significant, one can

test for differences among levels of one factor for fixed levels of the other two, i.e., simple, simple main effects. If the second order interaction is not significant, but at least one of the first order interactions is, hypotheses associated with simple main effects can be tested. These involve testing for differences among the levels of one factor for a fixed level of another factor and averaged across the levels of the remaining factor. This reasoning can be extended to higher order factorials.

Suppose the third order interaction is significant in a four factor MANOVA. One can consider simple second order interactions, i.e., three factor interactions for fixed levels of the fourth factor. Choose the significant simple second order interactions and consider simple, simple first order interactions, i.e., first order interactions for fixed levels of two factors. Finally, among the significant simple, simple first order interactions, consider simple, simple, simple main effects, i.e., differences among the levels of one factor for fixed levels of three factors. If the third order interaction is not significant and at least one of the second order interactions is significant, hypotheses associated with first order interactions for fixed levels of a factor collapsed across the levels of another can be considered. Among these significant findings one can test for differences among levels of one factor for fixed levels of two factors and collapsed across the levels of the remaining factor. If the second and third order

interactions are not significant, and at least one first order interaction is significant, hypotheses can be formulated to test differences among levels of one factor for fixed levels of another factor collapsed across the levels of two factors.

All of the above hypotheses are implied by the overall hypothesis of equality of population cell means for factorial designs in the context of the STP. In general, it is possible to test the joint nullity of any subset of contrasts among population means. In the sequel we will illustrate how this is done. First, the basic results for STPS in the context of the unrestricted full rank general linear multivariate model for population cell means are presented.

## Hypothesis Testing in the Full-Rank Multivariate Population Means Model

For ease of writing consider a two-way design with I classes in the A way of classification and J classes in the B way. Write the observational model for the kth experimental unit in the (i, j)-cell as

$$[1] \quad \underset{\sim}{y}_{ijk} = \underset{\sim}{\xi}_{ij} + \underset{\sim}{\varepsilon}_{ijk}$$
$$i = 1, 2, \cdots, I; \; j = 1, 2, \cdots, J; \; k = 1, 2, \cdots, K_{ij}$$

$\underset{\sim}{y}'_{ijk} = (y^1_{ijk}, y^2_{ijk}, \cdots, y^p_{ijk})$ is a p-variate vector of observations for the kth subject in the (i, j)-cell, $\underset{\sim}{\xi}'_{ijk} = (\xi^1_{ij}, \xi^2_{ij}, \cdots, \xi^p_{ij})$ is the p-variate vector of population cell means for the (i, j)-cell and $\underset{\sim}{\varepsilon}'_{ijk} = (\varepsilon^1_{ijk}, \varepsilon^2_{ijk}, \cdots, \varepsilon^p_{ijk})$ is a p-variate vector of random errors following a multivariate normal distribution with mean the zero vector and variance-covariance matrix

$\underset{\sim}{\Sigma}$ (p x p). Let n = IJ and $\sum_{i=1}^{I}\sum_{j=1}^{J}$ = N. Extension of the notation to designs with more than two factors should present no difficulty. A comprehensive discussion for a commonly occurring collection of experimental designs is given in Timm and Carlson (1975). In matrix notation, the model may be written as

$$[2] \qquad \underset{\sim}{Y}(N \times p) = \underset{\sim}{X}(N \times n)\ \underset{\sim}{\Xi}(n \times p) + \underset{\sim}{E}(N \times p)$$

where $\underset{\sim}{Y}(N \times p)$ has as rows N independent p-variate observations with common variance-covariance matrix $\underset{\sim}{\Sigma}(p \times p)$ and expectations such that $E[\underset{\sim}{Y}(N \times p)] = \underset{\sim}{X}(N \times n)\ \underset{\sim}{\Xi}(n \times p)$. $\underset{\sim}{\Xi}(n \times p)$ has as rows the n p-variate population mean vectors. $\underset{\sim}{X}(N \times n)$ is a matrix of full column rank consisting of 0's and 1's. $\underset{\sim}{E}(N \times p)$ has as rows N independent p-variate vectors of random errors with mean vector zero and common variance-covariance matrix $\underset{\sim}{\Sigma}$ (p x p).

Under this model any linear null hypothesis can be written in the form

$$[3] \quad H_i: \underset{\sim}{\phi}_i(c_i \times a_i) = \underset{\sim}{C}_i'(c_i \times n)\ \underset{\sim}{\Xi}(n \times p)\ \underset{\sim}{A}_i(p \times a_i) = \underset{\sim}{0}(c_i \times a_i)$$

for some $c_i$ x n matrix $\underset{\sim}{C}_i'$ of rank $c_i$ and p x $a_i$ matrix $\underset{\sim}{A}_i$ of rank $a_i$. Least squares estimates of the parametric function $\underset{\sim}{\phi}_i$ may be written as

$$[4] \qquad \hat{\underset{\sim}{\phi}}_i(c_i \times a_i) = \underset{\sim}{C}_i'(c_i \times n)\ \hat{\underset{\sim}{\Xi}}(n \times p)\ \underset{\sim}{A}_i(p \times a_i)$$

where

[5] $\hat{\underset{\sim}{\Xi}}(n \times p) = \underset{\sim}{D}^{-1}(n \times n)\ \underset{\sim}{X}'(n \times N)\ \underset{\sim}{Y}(N \times p)$

$\underset{\sim}{D}^{-1}$ is the inverse of $\underset{\sim}{X}'(n \times N)\ \underset{\sim}{X}(N \times n)$ and has as diagonal elements the reciprocals of the subclass frequencies. The rows of $\hat{\underset{\sim}{\Xi}}$ consist of n p-variate subclass mean vectors. If

[6] $\underset{\sim}{W}_i(c_i \times c_i) = \underset{\sim}{C}_i'(c_i \times n)\ \underset{\sim}{D}^{-1}(n \times n)\ \underset{\sim}{C}_i(N \times c_i)$

then the $\underset{\sim}{H}_i$ hypothesis sum of squares and cross products matrix (SSCP for short) is

[7] $\underset{\sim}{H}_i(a_i \times a_i) = \hat{\underset{\sim}{\phi}}_i'(a_i \times c_i)\ \underset{\sim}{W}_i^{-1}(c_i \times c_i)\ \hat{\underset{\sim}{\phi}}_i(c_i \times a_i)$

and the corresponding matrix of sums of squares and cross products for error is

[8] $\underset{\sim}{E}_i(a_i \times a_i) = \underset{\sim}{A}_i'(\underset{\sim}{Y}'\underset{\sim}{Y} - \underset{\sim}{Y}'\underset{\sim}{X}\underset{\sim}{D}^{-1}\underset{\sim}{X}'\underset{\sim}{Y})\ \underset{\sim}{A}_i$

Multiple criteria exist for testing the hypothesis in [3]. All criteria are based on some function of the eigenvalues of $\underset{\sim}{H}_i\underset{\sim}{E}_i^{-1}$ where $\underset{\sim}{H}_i$ and $\underset{\sim}{E}_i$ are given in [7] and [8], respectively. In this paper four criteria will be given and are based on the notion of an increasing root function which is fundamental to the theoretical basis underlying multivariate STPS. A full account has been given by Gabriel (1968). Let $\Psi$ denote an increasing root function. For our purposes we need only know that $\Psi$ represents some function of the eigenvalues of $\underset{\sim}{H}_i\underset{\sim}{E}_i^{-1}$.

Let $\lambda_1, \lambda_2, \cdots, \lambda_{s_i}$ denote the eigenvalues of $\underset{\sim}{H}_i\underset{\sim}{E}_i^{-1}$ in descending order of magnitude where $s_i = \min\{c_i, a_i\}$ and let

$$[9] \quad m_i = \frac{|\, a_i - c_i \,| - 1}{2} \text{ and } n_i = \frac{N - n - a_i - 1}{2}$$

Gabriel (1968) gives the following four increasing root functions for testing [3] in conjunction with the STP. They are

$$[10] \quad \Psi_1(H_i E_i^{-1}) = \lambda_1, \text{ i.e., Roy's Largest root}$$

$$\Psi_2(H_i E_i^{-1}) = \sum_{j=1}^{s_i} \lambda_j, \text{ i.e., Lawley-Hotelling trace}$$

$$\Psi_3(H_i E_i^{-1}) = \prod_{j=1}^{s_i} (1 + \lambda_j) - 1, \text{ i.e., this involves Wilk's } \Lambda$$

$$\Psi_4(H_i E_i^{-1}) = \frac{\sum_{j=1} \theta_j}{1 - \sum_{j=1}^{s_i} \theta_j} \text{ where } \theta_j = \frac{\lambda_j}{1 + \lambda_j}, \text{ i.e. this}$$

involves Pillai's trace.

Depending on the choice of test criterion, hypothesis $H_i$ in [3] is rejected at the $\alpha$-level of significance if

$$[11] \quad \Psi_1(H_i E_i^{-1}) > \frac{\theta^\alpha (s_i, m_i, n_i)}{1 - \theta^\alpha (s_i, m_i, n_i)} = \Psi_1^\alpha$$

$$\Psi_2(H_i E_i^{-1}) > \upsilon_o^\alpha (s_i, m_i, n_i) = \Psi_2^\alpha$$

$$\Psi_3(H_i E_i^{-1}) > \frac{1 - \upsilon^\alpha (a_i, c_i, N - n)}{\upsilon^\alpha (a_i, c_i, N - n)} = \Psi_3^\alpha$$

$$\Psi_4(H_i E_i^{-1}) > \frac{V^\alpha (s_i, m_i, n_i)}{1 - V^\alpha (s_i, m_i, n_i)} = \Psi_4^\alpha$$

Charts, tables and further information pertaining to obtaining $\theta^{\alpha}$, $\upsilon_o^{\alpha}$, $\upsilon^{\alpha}$, and $V^{\alpha}$ are given in Timm (1975). If $s_i = \min\{c_i, a_i\} = 1$, then $\Psi_1$, $\Psi_2$, $\Psi_3$ and $\Psi_4$ are equivalent to

$$[12] \qquad F_i = F_i(\underset{\sim}{H}_i\underset{\sim}{E}_i^{-1}) = \Psi_i(\underset{\sim}{H}_i\underset{\sim}{E}_i^{-1})\,\frac{n_i + 1}{m_i + 1}$$

which is distributed as $F_i(2m_i + 2, 2n_i + 2)$ when $H_i$ is true.

## Simultaneous Test Procedures

For the model [2] consider a family of hypotheses $H = \{H_i: i \in I\}$, I being the nonnegative integers, such that

$$[13] \qquad H_o: \underset{\sim}{\phi}_o(c_o \times a_i) = \underset{\sim}{C}_o'(c_o \times n)\,\underset{\sim}{\Xi}(n \times p)\,\underset{\sim}{A}_o(p \times a_o) = \underset{\sim}{0}$$

belongs to H and all other $H_i \in H$ are implied by $H_o$, i.e., for every $i \in I$ there exists matrices $\underset{\sim}{B}_i$ $(c_o \times c_i)$ and $\underset{\sim}{F}_i$ $(a_o \times a_i)$ such that

$$[14] \qquad \underset{\sim}{C}_i(n \times c_i) = \underset{\sim}{C}_o(n \times c_o)\,\underset{\sim}{B}_i(c_o \times c_i)$$

and

$$\underset{\sim}{A}_i(p \times a_i) = \underset{\sim}{A}_o(p \times a_o)\,\underset{\sim}{F}_i(a_o \times a_i)$$

The family H may consist of all linear hypotheses implied by $H_o$ or only of part of these hypotheses. An $\alpha$-level $\Psi$-STP for family H, where $\Psi$ is chosen from [10], is defined as the family of tests rejecting any $H_i \in H$ if and only if

$$[15] \qquad \Psi(\underset{\sim}{H}_i\underset{\sim}{E}_i^{-1}) > \Psi^{\alpha}$$

where $\Psi^{\alpha}$ is a constant satisfying

[16] $$P_{H_o}\{\Psi(H_o E_o^{-1}) > \Psi^{\alpha}\} = \alpha$$

The adjective "simultaneous" in STP indicates that all hypotheses $H_i \in H$ are tested simultaneously against the critical value $\Psi^{\alpha}$ and without the need of referring from one test to another. The probability of making one or more type I errors in all the tests of an $\alpha$-level STP is exactly $\alpha$ if $H_o$ holds. Otherwise, it is no greater than $\alpha$. If $H_i$ and $H_j$ are hypotheses in H and $H_i$ implies $H_j$, then the STP rejects $H_i$ whenever $H_j$ is rejected. Conversely, $H_j$ must be accepted when $H_i$ is accepted. This is referred to as coherence.

Coherence gives rise to two classes of hypotheses which can be helpful in summarizing the results of hypothesis testing associated with the STP. The classes of hypotheses are referred to as minimal and maximal rejected hypotheses. If $H_i \in H$ and is rejected and implies no other hypotheses in H, we say $H_i$ is a minimal rejected hypothesis. If $H_j \in H$ and is accepted and all hypotheses implying $H_j$ are rejected we say $H_j$ is a maximal accepted hypothesis. By virtue of coherence, any hypothesis is known to be rejected if, and only if, it implies a minimal rejected hypothesis, and accepted, if and only if, it is implied by a maximal accepted hypothesis (Gabriel, 1968). Among the multivariate STPS, Gabriel (1968) has shown that the STP associated with Roy's Largest root, $\Psi_1$, is consonant, the most parsimonious and the most resolvent

for any family containing all minimal hypotheses, i.e., those for which $c_i = a_i = 1$. A consonant family is one in which a non-minimal rejected hypothesis implies at least one rejected hypothesis. Among all one-root hypotheses, i.e., those for which $s_i = \min\{c_i, a_i\} = 1$, $\Psi_1$, $\Psi_2$, $\Psi_3$ and $\Psi_4$ yield identical decisions. $\Psi_1$, based on Roy's largest root, is most parsimonious since it has the least probability of a type I error and is said to be most resolvent since it rejects all hypotheses rejected by the other STPS and possibly more.

It is possible to construct confidence intervals for all those parametric functions in H having the form

$$[17] \qquad \phi_i = \underset{\sim}{c}_i'(1 \times n)\ \underset{\sim}{\Xi}(n \times p)\ \underset{\sim}{d}_i(p \times 1)$$

These correspond to minimal hypotheses and have the form

$$[18] \qquad \hat{\phi}_i \pm \sqrt{\Psi^{\alpha}\ \underset{\sim}{W}_i \underset{\sim}{E}_i}$$

where $\underset{\sim}{W}_i$ and $\underset{\sim}{E}_i$ are given in [6] and [8], respectively, and $\Psi^{\alpha}$ may be chosen from [11]. The shortest confidence intervals are those associated with Roy's Largest root. If $p = 1$, [18] reduces to the familiar Scheffé intervals. For a two-way ANOVA with I levels of A and J levels of B, [18] becomes

$$[19] \qquad \sum_{i-1}^{I} \sum_{j=1}^{J} c_{ij} \hat{\xi}_{ij} \pm \sqrt{(IJ - 1)\ F_{\alpha;\ IJ - 1,\ N - IJ}\ \frac{E}{N - IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{c_{ij}^2}{n_{ij}}}$$

where E is the within cells sum of squares.

The definition and summary of the salient properties of STPS necessary for their application has been given. As to

which STP to choose in practice a number of considerations must be made. The interested reader is referred to Gabriel (1968, 1969a) and a recent Monte Carlo study by Olson (1973, 1974).

Two-Way MANOVA with Interaction

The classical two-way MANOVA design, with an equal number of observations per cell is

$$[20] \qquad \underset{\sim}{y}_{ijk} = \underset{\sim}{\mu} + \underset{\sim}{\alpha}_i + \underset{\sim}{\beta}_i + \underset{\sim}{\gamma}_{ij} + \underset{\sim}{\varepsilon}_{ijk}$$
$$i = 1, \cdots, I;\ j = 1, \cdots, J;\ k = 1, \cdots, K$$
$$\underset{\sim}{\varepsilon}_{ijk} \sim IN(\underset{\sim}{0}, \underset{\sim}{\Sigma}).$$

The associated full rank model is given in [1] where $\underset{\sim}{\xi}_{ij}$ replaces $\underset{\sim}{\mu} + \underset{\sim}{\alpha}_i + \underset{\sim}{\beta}_j + \underset{\sim}{\gamma}_{ij}$ in [20] and $K_{ij} = K$ for all (i,j)-pairs. The matrix $\underset{\sim}{\Xi}$ in [2] has the form

$$[21] \qquad \underset{\sim}{\Xi}'(p \times IJ) = (\underset{\sim}{\xi}_{11}, \cdots, \underset{\sim}{\xi}_{1J};\ \underset{\sim}{\xi}_{21}, \cdots, \underset{\sim}{\xi}_{2J};\ \cdots;\ \underset{\sim}{\xi}_{I1}, \cdots, \underset{\sim}{\xi}_{IJ})$$

As overall hypothesis we take $H_o$ to be the equality of the IJ-population means, i.e.,

$$[22] \qquad H_o: \ \underset{\sim}{\xi}_{11} = \underset{\sim}{\xi}_{12} = \cdots = \underset{\sim}{\xi}_{1J} =$$
$$\underset{\sim}{\xi}_{21} = \underset{\sim}{\xi}_{22} = \cdots = \underset{\sim}{\xi}_{2J} =$$
$$\underset{\sim}{\xi}_{I1} = \underset{\sim}{\xi}_{I2} = \cdots = \underset{\sim}{\xi}_{IJ}$$

In terms of [13], choose $\underset{\sim}{C}_o'(IJ - 1 \times IJ) = [\underset{\sim}{I}, -\underset{\sim}{1}]$ where $\underset{\sim}{I}$ is an identity matrix of order IJ - 1 and $\underset{\sim}{1}$ is a column matrix with IJ - 1 ones and $\underset{\sim}{A}_o = \underset{\sim}{I}_p$ where $\underset{\sim}{I}_p$ is an identity matrix of order p where p is the number of dependent variables in the

analysis. The hypothesis of equality of the I population row means can be written

$$[23] \qquad H_1: \sum_{j=1}^{J} \underset{\sim}{\xi}_{1j} = \sum_{j=1}^{J} \underset{\sim}{\xi}_{2j} = \cdots = \sum_{j=1}^{J} \underset{\sim}{\xi}_{IJ}$$

or

$$\underset{\sim}{\xi}_{1\cdot} = \underset{\sim}{\xi}_{2\cdot} = \cdots = \underset{\sim}{\xi}_{I\cdot}$$

In terms of [3] choose $\underset{\sim}{A}_1 = \underset{\sim}{I}_p$ and

$$[24] \qquad \underset{\sim}{C}_1'(I - 1 \times IJ) = \underset{\sim}{C}(I - 1 \times I) \otimes \frac{\underset{\sim}{1}'}{\sqrt{J}} (1 \times J)$$

where $\underset{\sim}{C}$ is a matrix having as rows I - 1 independent or orthogonal contrasts among the population row means in [23] and $\underset{\sim}{1}'$ is a row matrix having all ones. $\otimes$ denotes the direct or Kronecker product of two matrices.

The hypothesis of equality of the J population column means is

$$[25] \qquad H_2: \sum_{i=1}^{I} \underset{\sim}{\xi}_{i1} = \sum_{i=1}^{I} \underset{\sim}{\xi}_{i2} = \cdots = \sum_{i=1}^{I} \underset{\sim}{\xi}_{iJ}$$

or

$$\underset{\sim}{\xi}_{\cdot 1} = \underset{\sim}{\xi}_{\cdot 2} = \cdots = \underset{\sim}{\xi}_{\cdot J}$$

In terms of [3] choose $\underset{\sim}{A}_2 = \underset{\sim}{I}_p$ and

$$[26] \qquad \underset{\sim}{C}_2'(J - 1 \times IJ) = \frac{\underset{\sim}{1}'}{\sqrt{I}} (1 \times I) \otimes \underset{\sim}{D}(J - 1 \times J)$$

where $\underset{\sim}{D}$ is a matrix having as rows J - 1 independent or orthogonal contrasts among the population column means in [25].

The hypothesis of no interaction can be written

[27] $H_3$: $\underset{\sim}{\xi}_{ij} - \underset{\sim}{\xi}_{i'j} - \underset{\sim}{\xi}_{ij'} + \underset{\sim}{\xi}_{i'j'} = \underset{\sim}{0}$

for all i, j, i' and j'. [27] indicates that the difference in population cell centroids for i and i' at j should be the same as the difference at j' if no interaction is present in the data. The formulation of [27] in terms of [3] can be effected by choosing $\underset{\sim}{A}_3 = \underset{\sim}{I}_p$ and

[28] $\underset{\sim}{C}_3'((I - 1)(J - 1) \times IJ) = \underset{\sim}{C}(I - 1 \times I) \otimes \underset{\sim}{D}(J - 1 \times J)$

where $\underset{\sim}{C}$ and $\underset{\sim}{D}$ are given in [24] and [26]. By choosing $\underset{\sim}{C}$ and $\underset{\sim}{D}$ to be orthonormal, i.e., $\underset{\sim}{C}'\underset{\sim}{C} = \underset{\sim}{I}$ and $\underset{\sim}{D}'\underset{\sim}{D} = \underset{\sim}{I}$ it follows that

[29] $\underset{\sim}{C}_1'\underset{\sim}{C}_1 = \underset{\sim}{I}(I - 1 \times I - 1)$

$\underset{\sim}{C}_2'\underset{\sim}{C}_2 = \underset{\sim}{I}(J - 1 \times J - 1)$

$\underset{\sim}{C}_3'\underset{\sim}{C}_3 = \underset{\sim}{I}((I - 1)(J - 1) \times (I - 1)(J - 1))$

$\underset{\sim}{C}_1'\underset{\sim}{C}_2 = \underset{\sim}{0}(I - 1 \times J - 1)$

$\underset{\sim}{C}_1'\underset{\sim}{C}_3 = \underset{\sim}{0}(I - 1 \times (I - 1)(J - 1))$

$\underset{\sim}{C}_2'\underset{\sim}{C}_3 = \underset{\sim}{0}(J - 1 \times (I - 1)(J - 1))$

It follows from [29] that by proper choice of $\underset{\sim}{C}$ and $\underset{\sim}{D}$ the hypothesis SSCP matrices associated with [24], [26] and [28], viz., $\underset{\sim}{H}_1$, $\underset{\sim}{H}_2$ and $\underset{\sim}{H}_3$ satisfy

[30] $\underset{\sim}{H}_1 + \underset{\sim}{H}_2 + \underset{\sim}{H}_3 = \underset{\sim}{B}$

where $\underset{\sim}{B}$ represents the between groups SSCP matrix with IJ - 1 degrees of freedom.

Simple Row and Column Effects Hypotheses

Whenever a significant interaction occurs, it is a signal to an experimenter that interpretation of tests of row and column effects must be qualified. Whether the tests associated with [23] and [25] are significant or not, the rejection of the interaction hypothesis [27] is a signal to look for differences among row means for specific columns or differences among column means for specific rows. The former will be called simple row effects and the latter simple column effects.

The hypothesis associated with the simple row effects of factor A at the jth level of factor B can be written

$$[31] \quad H_{j+3}: \ \underset{\sim}{\xi}_{1j} = \underset{\sim}{\xi}_{2j} = \cdots = \underset{\sim}{\xi}_{Ij}$$

for $j = 1, 2, \cdots, J$. Let $\underset{\sim}{1}'_j$ denote a 1 x J row matrix with one in the jth column and zeroes elsewhere. Let $\underset{\sim}{G}_j$ (I - 1 x I) be a contrast matrix for the simple row effects of factor A at the jth level of B. The J simple row effects hypotheses in [31] can be formulated in terms of [3] by choosing $\underset{\sim}{A}_{j+3} = \underset{\sim}{I}_p$ and

$$[32] \quad \underset{\sim}{C}'_{j+3}(I - 1 \times IJ) = \underset{\sim}{G}_j(I - I \times I) \otimes \underset{\sim}{1}'_j(1 \times J)$$

for $j = 1, 2, \cdots, J$. Note that for $j \neq \ell$

$$[33] \quad \underset{\sim}{C}'_{j+3}\underset{\sim}{C}_{\ell+3} = (\underset{\sim}{G}_j \times \underset{\sim}{1}'_j)(\underset{\sim}{G}'_\ell \otimes \underset{\sim}{1}_\ell)$$

$$= (\underset{\sim}{G}_j\underset{\sim}{G}'_\ell) \otimes \underset{\sim}{1}'_j\underset{\sim}{1}_\ell = \underset{\sim}{I} \otimes 0$$

$$= \underset{\sim}{0}(I - 1 \times I - 1)$$

Note also that

$$[34] \quad \underset{\sim}{C}'_{j+3}\underset{\sim}{C}_2 = \underset{\sim}{0}(I - 1 \times J - 1)$$

From [33] the J simple row effects hypotheses of factor A are mutually orthogonal and involve J(I - 1) degrees of freedom. [34] implies the simple row effects contrasts are orthogonal to the contrasts associated with testing column effects. Thus the SSCP matrices associated with simple row effects for factor A at the J levels of factor B along with the SSCP matrices associated with column effects exhausts the between groups variation. Since the hypothesis SSCP matrices associated with [31] are $\underset{\sim}{H}_4, \underset{\sim}{H}_5, \cdots \underset{\sim}{H}_{J+3}$ we have

$$[35] \quad \underset{\sim}{H}_2 + \sum_{j=1}^{J} \underset{\sim}{H}_{j+3} = \underset{\sim}{B}.$$

Comparing [35] with [30], we have

$$[36] \quad \sum_{j=1}^{J} \underset{\sim}{H}_{j+3} = \underset{\sim}{H}_1 + \underset{\sim}{H}_3$$

i.e., the sum of the hypothesis SSCP matrices associated with the simple row effects of factor A for the J levels of factor B is equal to the sum of the hypothesis SSCP matrices associated

with testing row differences and interaction effects. These results have been stated without proof for the univariate case (p = 1) by Kirk (1968) and Winer (1971). Prior to this article, the author knows of no extension, proof or application of the results to the multivariate case.

The hypothesis associated with the simple column effects of factor B at the ith level of factor A can be written

$$[37] \quad H_{i+J+3}: \ \underset{\sim}{\xi}_{i1} = \underset{\sim}{\xi}_{i2} = \cdots = \underset{\sim}{\xi}_{iJ}$$

for i = 1, 2, · · ·, I. Let $\underset{\sim}{1}_i'$ denote a 1 x I row matrix with one in the ith column and zeroes elsewhere. Let $\underset{\sim}{T}_i$ (J - 1 x J) be a contrast matrix for the simple column effects of factor B at the ith level of A. The I simple column effects hypotheses in [37] can be formulated in terms of [3] by choosing $\underset{\sim}{A}_{i+J+3} = \underset{\sim}{I}_p$ and

$$[38] \quad \underset{\sim}{C}_{i+J+3}'(J - 1 \times IJ) = \underset{\sim}{1}_i'(1 \times I) \otimes \underset{\sim}{T}_i(J - 1 \times J)$$

for i = 1, 2, · · ·, I. Note that for $i \neq \ell$

$$[39] \quad \underset{\sim}{C}_{i+J+3}'\underset{\sim}{C}_{\ell+J+3} = (\underset{\sim}{1}_i'\underset{\sim}{1}_\ell) \otimes \underset{\sim}{T}_i\underset{\sim}{T}_\ell' = \underset{\sim}{0}(J - 1 \times J - 1)$$

and also that

$$[40] \quad \underset{\sim}{C}_{i+J+3}'\underset{\sim}{C}_1 = (\underset{\sim}{1}_i' \otimes \underset{\sim}{T}_i)(\underset{\sim}{C}' \otimes \underset{\sim}{1}) = (\underset{\sim}{1}_i'\underset{\sim}{C}') \otimes (\underset{\sim}{T}_i\underset{\sim}{1}) = \underset{\sim}{0}(J - 1 \times I - 1)$$

From [39] the I simple column effects hypotheses of factor B are mutually orthogonal and involve I(J - 1) degrees of freedom. [40] implies the simple column effects contrasts are orthogonal to the contrasts associated with testing row effects. Thus the SSCP matrices associated with simple column effects for factor B at the I levels of factor A along with the SSCP matrices associated with row effects exhausts the between group variation. Since the hypothesis SSCP matrices associated with [37] are $\underset{\sim}{H}_{J+4}, \underset{\sim}{H}_{J+5}, \cdots, \underset{\sim}{H}_{I+J+3}$ we have

$$[41] \quad \underset{\sim}{H}_1 + \sum_{i=1}^{I} \underset{\sim}{H}_{i+J+3} = \underset{\sim}{B}$$

Comparing [41] with [30], we have

$$[42] \quad \sum_{i=1}^{I} \underset{\sim}{H}_{i+J+3} = \underset{\sim}{H}_2 + \underset{\sim}{H}_3$$

i.e., the sum of the hypothesis SSCP matrices associated with the simple column effects of factor B for the I levels of factor A is equal to the sum of the hypothesis SSCP matrices associated with testing column effects and interaction effects. [36] and [42] can serve as computational checks and have been stated for the univariate case by Kirk (1968) and Winer (1971).

If either [31] for some j or [37] for some i is rejected after having rejected [27], an experimenter may be interested in making comparisons among means for these simple row and column effects. The desired comparisons can be constructed to make up the rows of $\underset{\sim}{G}_j$ in [32] and $\underset{\sim}{T}_i$ in [38], respectively.

These hypotheses together with those in [23], [25], [27], [31] and [33] make up a family implied by $H_0$ in [22]. There are many other hypotheses in the family implied by [22] and the STP allows the experimenter to explore his data for the sources of significance while maintaining control of his type I error. Depending on the data it may be possible for a researcher to select a stronger overall hypothesis, which will imply fewer hypotheses including those of interest, and thus increase his power for the family of tests under consideration.

### Three-Way MANOVA with Interaction

The full rank model for the three-way design with interaction and an equal number of observations per cell is

$$[43] \quad \underset{\sim}{y}_{ijkm} = \underset{\sim}{\xi}_{ijkm} + \underset{\sim}{\varepsilon}_{ijkm}$$
$$i = 1, \cdots, I;\ j = 1, \cdots, J;\ k = 1, \cdots, K;$$
$$m = 1, \cdots, M$$

where $\underset{\sim}{\xi}_{ijk} = \underset{\sim}{\mu} + \underset{\sim}{\alpha}_i + \underset{\sim}{\beta}_j + \underset{\sim}{\gamma}_k + (\underset{\sim}{\alpha\beta})_{ij} + (\underset{\sim}{\alpha\gamma})_{ik} + (\underset{\sim}{\beta\gamma})_{jk} + (\underset{\sim}{\alpha\beta\gamma})_{ijk}$ for the classical model. The matrix $\underset{\sim}{\Xi}$ has the form

$$[44] \quad \underset{\sim}{\Xi}'(p \times IJK) = (\underset{\sim}{\xi}_{111}, \cdots, \underset{\sim}{\xi}_{11K}; \cdots; \underset{\sim}{\xi}_{1J1}, \cdots, \underset{\sim}{\xi}_{1JK};$$
$$\underset{\sim}{\xi}_{211}, \cdots, \underset{\sim}{\xi}_{21K}; \cdots; \underset{\sim}{\xi}_{2J1}, \cdots, \underset{\sim}{\xi}_{2JK};$$
$$\underset{\sim}{\xi}_{I11}, \cdots, \underset{\sim}{\xi}_{I1K}; \cdots; \underset{\sim}{\xi}_{IJ1}, \cdots, \underset{\sim}{\xi}_{IJK})$$

The hypotheses associated with the usual three-way MANOVA in terms of the $\underset{\sim}{\xi}_{ijk}$'s are

[45] $H_1^A$: all $\underset{\sim}{\xi}_{i\cdot\cdot}$'s are equal

$H_2^B$: all $\underset{\sim}{\xi}_{\cdot j\cdot}$'s are equal

$H_3^C$: all $\underset{\sim}{\xi}_{\cdot\cdot k}$'s are equal

$H_4^{AB}$: $\underset{\sim}{\xi}_{ij\cdot} - \underset{\sim}{\xi}_{i'j\cdot} - \underset{\sim}{\xi}_{ij'\cdot} + \underset{\sim}{\xi}_{i'j'\cdot} = \underset{\sim}{0}$ for all i, j, i' and j'

$H_5^{AC}$: $\underset{\sim}{\xi}_{i\cdot k} - \underset{\sim}{\xi}_{i'\cdot k} - \underset{\sim}{\xi}_{i\cdot k'} + \underset{\sim}{\xi}_{i'\cdot k'} = \underset{\sim}{0}$ for all i, k, i' and k'

$H_6^{BC}$: $\underset{\sim}{\xi}_{\cdot jk} - \underset{\sim}{\xi}_{\cdot j'k} - \underset{\sim}{\xi}_{\cdot jk'} + \underset{\sim}{\xi}_{\cdot j'k'} = \underset{\sim}{0}$ for all j, k, j' and k'

$H_7^{ABC}$: $\underset{\sim}{\xi}_{ijk} - \underset{\sim}{\xi}_{i'jk} - \underset{\sim}{\xi}_{ij'k} + \underset{\sim}{\xi}_{i'j'k} - \underset{\sim}{\xi}_{ijk'} + \underset{\sim}{\xi}_{i'jk'} + \underset{\sim}{\xi}_{ij'k'} - \underset{\sim}{\xi}_{i'j'k'} = \underset{\sim}{0}$ for all i, j, k, i', j' and k'

In order to formulate the hypotheses in [45] in terms of [3], choose $\underset{\sim}{A}_i = \underset{\sim}{I}_p$ and construct contrast matrices $\underset{\sim}{C}(I - 1 \times I)$, $\underset{\sim}{D}(J - 1 \times J)$ and $\underset{\sim}{E}(K - 1 \times K)$ such that $\underset{\sim}{C}\,\underset{\sim}{C}' = \underset{\sim}{I}$, $\underset{\sim}{D}\,\underset{\sim}{D}' = \underset{\sim}{I}$ and $\underset{\sim}{E}'\underset{\sim}{E} = \underset{\sim}{I}$. Letting $\underset{\sim}{1}'$ denote an appropriate row matrix of ones and utilizing properties of the Kronecker product the $\underset{\sim}{C}_i$ matrices associated with the hypotheses in [45] can be written

$$[46]\quad \underset{\sim}{C}_1'(I - 1 \times IJK) = \underset{\sim}{C}(I - 1 \times I) \otimes \underset{\sim}{1}'(1 \times JK)$$

$$\underset{\sim}{C}_2'(J - 1 \times IJK) = \underset{\sim}{1}'(1 \times I) \otimes \underset{\sim}{D}(J - 1 \times J) \otimes \underset{\sim}{1}'(1 \times K)$$

$$\underset{\sim}{C}_3'(K - 1 \times IJK) = \underset{\sim}{1}'(1 \times IJ) \otimes \underset{\sim}{E}(K - 1 \times K)$$

$$\underset{\sim}{C}_4'((I - 1)(J - 1) \times IJK) = \underset{\sim}{C}(I - 1 \times I) \otimes \underset{\sim}{D}(J - 1 \times J) \otimes \underset{\sim}{1}'(1 \times K)$$

$$\underset{\sim}{C}_5'((I - 1)(K - 1) \times IJK) = \underset{\sim}{C}(I - 1 \times I) \otimes \underset{\sim}{1}'(1 \times J) \otimes \underset{\sim}{E}(K - 1 \times K)$$

$$\underset{\sim}{C}_6'((J - 1)(K - 1) \times IJK) = \underset{\sim}{1}'(1 \times I) \otimes \underset{\sim}{D}(J - 1 \times J) \otimes \underset{\sim}{E}(K - 1 \times K)$$

$$\underset{\sim}{C}_7'((I - 1)(J - 1)(I - 1) \times IJK) = \underset{\sim}{C}(I - 1 \times I) \otimes \underset{\sim}{D}(J - 1 \times J) \otimes \underset{\sim}{E}(K - 1 \times K)$$

By choosing $\underset{\sim}{C}$, $\underset{\sim}{D}$ and $\underset{\sim}{E}$ orthonormal and normalizing the unit row matrices it follows that $\underset{\sim}{C}_i'\underset{\sim}{C}_j$ will be equal to an identity matrix of appropriate order for $i = j$ and a matrix of zeroes when $i \neq j$. Thus it follows that the hypothesis SSCP matrices associated with [46], viz., $\underset{\sim}{H}_1$, $\underset{\sim}{H}_2$, $\underset{\sim}{H}_3$, $\underset{\sim}{H}_4$, $\underset{\sim}{H}_5$, $\underset{\sim}{H}_6$ and $\underset{\sim}{H}_7$ satisfy

$$[47]\quad \sum_{i=1}^{7} \underset{\sim}{H}_i = \underset{\sim}{B}$$

where $\underset{\sim}{B}$ represents the between groups SSCP matrix with IJK - 1 degrees of freedom.

Analysis of a Three-Factor Interaction

If the three-factor interaction is found to be significant, tests of simple interaction effects as well as simple simple A, B and C effects can be formulated. Simple interaction effects hypotheses involve testing for first order interactions for a fixed level of the other factor. Simple simple effects hypotheses involve testing for differences among the levels of one factor for fixed levels of the other factors.

The hypotheses associated with simple interaction effects can be written

[48] $H^{AB}_{c_k}$: $\underset{\sim}{\xi}_{ijk} - \underset{\sim}{\xi}_{i'jk} - \underset{\sim}{\xi}_{ij'k} + \underset{\sim}{\xi}_{i'j'k} = \underset{\sim}{0}$ for all i, j, i', j' and fixed k.

$H^{AC}_{b_j}$: $\underset{\sim}{\xi}_{ijk} - \underset{\sim}{\xi}_{i'jk'} - \underset{\sim}{\xi}_{ijk'} + \underset{\sim}{\xi}_{i'jk'} = \underset{\sim}{0}$ for all i, k, i', k' and fixed j.

$H^{BC}_{a_i}$: $\underset{\sim}{\xi}_{ijk} - \underset{\sim}{\xi}_{ij'k} - \underset{\sim}{\xi}_{ijk} + \underset{\sim}{\xi}_{ij'k'} = \underset{\sim}{0}$ for all j, k, j', k' and fixed i.

[48] can be written in terms of [3] by choosing $\underset{\sim}{A}_i = \underset{\sim}{I}_p$ and

[49] $\underset{\sim}{C}'_{c_k}((I - 1)(J - 1) \times IJK) = \underset{\sim}{C}(I - 1 \times I) \otimes \underset{\sim}{D}(J - 1 \times J) \otimes \underset{\sim}{1}'_k(1 \times K)$

$\underset{\sim}{C}'_{b_j}((I - 1)(K - 1) \times IJK) = \underset{\sim}{C}(I - 1 \times I) \otimes \underset{\sim}{i}'_j(1 \times J) \otimes \underset{\sim}{E}(K - 1 \times K)$

$$\underset{\sim}{C}'_{a_i}\,((J-1)(K-1) \times IJK) = \underset{\sim}{1}'_i(1 \times I) \otimes \underset{\sim}{D}(J-1 \times J) \otimes \underset{\sim}{E}(K-1 \times K)$$

Using properties of Kronecker products it follows that

[50] $\underset{\sim}{C}'_{c_k}\underset{\sim}{C}_\ell = \underset{\sim}{0}$ for $\ell = 1, 2, 3, 5, 6$ and all $k$

$\underset{\sim}{C}'_{b_j}\underset{\sim}{C}_\ell = \underset{\sim}{0}$ for $\ell = 1, 2, 3, 4, 6$ and all $j$

$\underset{\sim}{C}'_{a_i}\underset{\sim}{C}_\ell = \underset{\sim}{0}$ for $\ell = 1, 2, 3, 4, 5$ and all $i$

Denoting the hypothesis SSCP matrices associated with [48] by $\underset{\sim}{H}^{AB}_{c_k}$, $\underset{\sim}{H}^{AC}_{b_j}$ and $\underset{\sim}{H}^{BC}_{a_i}$ and using [50] it follows that

[51] $$\sum_{k=1}^{K} \underset{\sim}{H}^{AB}_{c_k} + \underset{\sim}{H}_1 + \underset{\sim}{H}_2 + \underset{\sim}{H}_3 + \underset{\sim}{H}_5 + \underset{\sim}{H}_6 = \underset{\sim}{B}$$

$$\sum_{j=1}^{J} \underset{\sim}{H}^{AC}_{b_j} + \underset{\sim}{H}_1 + \underset{\sim}{H}_2 + \underset{\sim}{H}_3 + \underset{\sim}{H}_4 + \underset{\sim}{H}_6 = \underset{\sim}{B}$$

$$\sum_{i=1}^{I} \underset{\sim}{H}^{BC}_{a_i} + \sum_{j=1}^{5} \underset{\sim}{H}_j = \underset{\sim}{B}$$

Comparing [51] with [47] we have

[52] $$\sum_{k=1}^{K} \underset{\sim}{H}^{AB}_{c_k} = \underset{\sim}{H}_4 + \underset{\sim}{H}_7$$

$$\sum_{j=1}^{J} \underset{\sim}{H}^{AC}_{b_j} = \underset{\sim}{H}_5 + \underset{\sim}{H}_7$$

$$\sum_{i=1}^{I} \underset{\sim}{H}^{BC}_{a_i} = \underset{\sim}{H}_6 + \underset{\sim}{H}_7$$

In words, the sum of the simple interaction hypothesis SSCP matrices for two factors across the levels of a third factor equals the overall interaction hypothesis SSCP matrix for the two factors plus the second order interaction hypothesis SSCP matrix.

The hypotheses associated with simple simple effects are

[53] $H^{A}_{bc_{jk}}$: $\underset{\sim}{\xi}_{1jk} = \underset{\sim}{\xi}_{2jk} = \cdots = \underset{\sim}{\xi}_{Ijk}$ for fixed (j, k)

$H^{B}_{ac_{ik}}$: $\underset{\sim}{\xi}_{i1k} = \underset{\sim}{\xi}_{i2k} = \cdots = \underset{\sim}{\xi}_{iJk}$ for fixed (i, k)

$H^{C}_{ab_{ij}}$: $\underset{\sim}{\xi}_{ij1} = \underset{\sim}{\xi}_{ij2} = \cdots = \underset{\sim}{\xi}_{ijK}$ for fixed (i, j)

Let $\underset{\sim}{G}_{jk}(I - 1 \times I)$, $\underset{\sim}{G}_{ik}(J - 1 \times J)$ and $\underset{\sim}{G}_{ij}(K - 1 \times K)$ denote contrast matrices. Then the hypotheses in [53] can be written in terms of [3] by choosing $\underset{\sim}{A}_i = \underset{\sim}{I}_p$ and

[54] $\underset{\sim}{C}'_{jk}(I - 1 \times IJK) = \underset{\sim}{G}_{jk}(I - 1 \times I) \otimes \underset{\sim}{1}'_j(1 \times J)$

$\otimes \underset{\sim}{1}'_k(1 \times K)$ for fixed (j, k)

$\underset{\sim}{C}'_{ik}(J - 1 \times IJK) = \underset{\sim}{1}'_i(1 \times I) \otimes \underset{\sim}{G}_{ik}(J - 1 \times J)$

$\otimes \underset{\sim}{1}'_k(1 \times K)$ for fixed (i, k)

$\underset{\sim}{C}'_{ij}(K - 1 \times IJK) = \underset{\sim}{1}'_i(1 \times I) \otimes \underset{\sim}{1}'_j(1 \times J)$

$\otimes \underset{\sim}{G}_{ij}(K - 1 \times K)$ for fixed (i, j)

Using properties of Kronecker products, it can be shown that

[55] $\underset{\sim}{C}'_{jk}\underset{\sim}{C}_{\ell} = 0$ for $\ell = 2, 3, 6$ and all $(j, k)$

$\underset{\sim}{C}'_{ik}\underset{\sim}{C}_{\ell} = 0$ for $\ell = 1, 3, 5$ and all $(i, k)$

$\underset{\sim}{C}'_{ij}\underset{\sim}{C}_{\ell} = 0$ for $\ell = 1, 2, 4$ and all $(i, j)$

Denoting the hypothesis SSCP matrices associated with [53] by $\underset{\sim}{H}^{A}_{jk}$, $\underset{\sim}{H}^{B}_{ik}$ and $\underset{\sim}{H}^{C}_{ij}$ and invoking [55] we have

[56] $$\sum_{j=1}^{J}\sum_{k=1}^{K}\underset{\sim}{H}^{A}_{ji} + \underset{\sim}{H}_2 + \underset{\sim}{H}_3 + \underset{\sim}{H}_6 = \underset{\sim}{B}$$

$$\sum_{i=1}^{I}\sum_{k=1}^{K}\underset{\sim}{H}^{B}_{ik} + \underset{\sim}{H}_1 + \underset{\sim}{H}_3 + \underset{\sim}{H}_5 = \underset{\sim}{B}$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\underset{\sim}{H}^{C}_{ij} + \underset{\sim}{H}_1 + \underset{\sim}{H}_2 + \underset{\sim}{H}_4 = \underset{\sim}{B}$$

where B is the between groups SSCP with IJK - 1 degrees of freedom. Comparing [56] with [47] we have

[57] $$\sum_{j=1}^{J}\sum_{k=1}^{K}\underset{\sim}{H}^{A}_{jk} = \underset{\sim}{H}_1 + \underset{\sim}{H}_4 + \underset{\sim}{H}_5 + \underset{\sim}{H}_7$$

$$\sum_{i=1}^{I}\sum_{k=1}^{K}\underset{\sim}{H}^{B}_{ik} = \underset{\sim}{H}_2 + \underset{\sim}{H}_4 + \underset{\sim}{H}_6 + \underset{\sim}{H}_7$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\underset{\sim}{H}^{C}_{ij} = \underset{\sim}{H}_3 + \underset{\sim}{H}_5 + \underset{\sim}{H}_6 + \underset{\sim}{H}_7$$

In words, the sum of the hypothesis SSCP matrices associated with testing the simple simple effects of a factor is equal to the sum of the hypothesis SSCP matrices associated with the factor, the first order interactions with the factor and the second order interaction, respectively.

## Simple Effects in Three-Way MANOVA

If the second order interaction is not significant, but at least one of the first order interactions is significant, attention can be focused on simple effects hypotheses. These hypotheses are analogous to those in a two-way MANOVA, the difference being that in a three-way MANOVA we explore for differences among the levels of one factor for a fixed level of another factor after having collapsed over the levels of the third factor. The hypotheses associated with simple effects for the model in [43] are

[58] $H^A_{b_j}$: $\underset{\sim}{\xi}_{ij\cdot}$'s are equal for all i and fixed j

$H^A_{c_k}$: $\underset{\sim}{\xi}_{i\cdot k}$'s are equal for all i and fixed k

$H^B_{a_i}$: $\underset{\sim}{\xi}_{ij\cdot}$'s are equal for all j and fixed i

$H^B_{c_k}$: $\underset{\sim}{\xi}_{\cdot jk}$'s are equal for all j and fixed k

$H^C_{a_i}$: $\underset{\sim}{\xi}_{i\cdot k}$'s are equal for all k and fixed i

$H^C_{b_j}$: $\underset{\sim}{\xi}_{\cdot jk}$'s are equal for all k and fixed j

The hypotheses in [58] can be written in terms of [3] by choosing $\underset{\sim}{A}_i = \underset{\sim}{I}_p$ and forming

[59] $\underset{\sim}{C}'_{b_j}(I - 1 \times IJK) = \underset{\sim}{G}_j(I - 1 \times I) \otimes \underset{\sim}{1}'_j(1 \times J) \otimes \underset{\sim}{1}'(1 \times K)$

$\underset{\sim}{C}'_{c_k}(I - 1 \times IJK) = \underset{\sim}{G}_k(I - 1 \times I) \otimes \underset{\sim}{1}'(1 \times J) \otimes \underset{\sim}{1}'_k(1 \times K)$

$$\underset{\sim}{C}'_{a_i}(J - 1 \times IJK) = \underset{\sim}{1}'_i(1 \times I) \otimes \underset{\sim}{D}_i(J - 1 \times J) \otimes \underset{\sim}{1}'(1 \times K)$$

$$\underset{\sim}{C}'_{c_k}(J - 1 \times IJK) = \underset{\sim}{1}'(1 \times I) \otimes \underset{\sim}{D}_k(J - 1 \times J) \otimes \underset{\sim}{1}'_k(1 \times K)$$

$$\underset{\sim}{C}'_{a_i}(K - 1 \times IJK) = \underset{\sim}{1}_i(1 \times I) \otimes \underset{\sim}{1}(1 \times J) \otimes \underset{\sim}{E}_i(K - 1 \times K)$$

$$\underset{\sim}{C}'_{b_j}(K - 1 \times IJK) = \underset{\sim}{1}'(1 \times I) \otimes \underset{\sim}{1}'_j(1 \times J) \otimes \underset{\sim}{E}_j(K - 1 \times K)$$

where the $\underset{\sim}{G}$, $\underset{\sim}{D}$ and $\underset{\sim}{E}$ matrices are contrast matrices. Using properties of Kronecker products we have that

[60] $$\underset{\sim}{C}'_{b_j}\underset{\sim}{C}_\ell = \underset{\sim}{0} \text{ for } \ell = 2, 3, 5, 6, 7 \text{ and all } j$$

$$\underset{\sim}{C}'_{c_k}\underset{\sim}{C}_\ell = \underset{\sim}{0} \text{ for } \ell = 2, 3, 4, 6, 7 \text{ and all } k$$

$$\underset{\sim}{C}'_{a_i}\underset{\sim}{C}_\ell = \underset{\sim}{0} \text{ for } \ell = 1, 3, 5, 6, 7 \text{ and all } i$$

$$\underset{\sim}{C}'_{c_k}\underset{\sim}{C}_\ell = \underset{\sim}{0} \text{ for } \ell = 1, 3, 4, 5, 7 \text{ and all } k$$

$$\underset{\sim}{C}'_{a_i}\underset{\sim}{C}_\ell = \underset{\sim}{0} \text{ for } \ell = 1, 2, 4, 6, 7 \text{ and all } i$$

$$\underset{\sim}{C}'_{b_j}\underset{\sim}{C}_\ell = \underset{\sim}{0} \text{ for } \ell = 1, 2, 4, 5, 7 \text{ and all } j$$

Denoting the SSCP matrices associated with [58] by $\underset{\sim}{H}^A_{b_j}$, $\underset{\sim}{H}^A_{c_k}$, etc. and using [60] it follows that

[61] $$\sum_{j=1}^{J} \underset{\sim}{H}^A_{b_j} + \underset{\sim}{H}_2 + \underset{\sim}{H}_3 + \underset{\sim}{H}_5 + \underset{\sim}{H}_6 + \underset{\sim}{H}_7 = \underset{\sim}{B}$$

$$\sum_{k=1}^{K} \underset{\sim}{H}^A_{c_k} + \underset{\sim}{H}_2 + \underset{\sim}{H}_3 + \underset{\sim}{H}_4 + \underset{\sim}{H}_6 + \underset{\sim}{H}_7 = \underset{\sim}{B}$$

$$\sum_{i=1}^{I} \underset{\sim}{H}^B_{a_i} + \underset{\sim}{H}_1 + \underset{\sim}{H}_3 + \underset{\sim}{H}_5 + \underset{\sim}{H}_6 + \underset{\sim}{H}_7 = \underset{\sim}{B}$$

$$\sum_{k=1}^{K} \underset{\sim}{H}^{B}_{c_k} + \underset{\sim}{H}_1 + \underset{\sim}{H}_3 + \underset{\sim}{H}_4 + \underset{\sim}{H}_5 + \underset{\sim}{H}_7 = \underset{\sim}{B}$$

$$\sum_{i=1}^{I} \underset{\sim}{H}^{C}_{a_i} + \underset{\sim}{H}_1 + \underset{\sim}{H}_2 + \underset{\sim}{H}_4 + \underset{\sim}{H}_6 + \underset{\sim}{H}_7 = \underset{\sim}{B}$$

$$\sum_{j=1}^{J} \underset{\sim}{H}^{C}_{b_j} + \underset{\sim}{H}_1 + \underset{\sim}{H}_2 + \underset{\sim}{H}_4 + \underset{\sim}{H}_5 + \underset{\sim}{H}_7 = \underset{\sim}{B}$$

Comparing [61] with [47], we have

[62]
$$\sum_{j=1}^{J} \underset{\sim}{H}^{A}_{b_j} = \underset{\sim}{H}_1 + \underset{\sim}{H}_4$$

$$\sum_{k=1}^{K} \underset{\sim}{H}^{A}_{c_k} = \underset{\sim}{H}_1 + \underset{\sim}{H}_5$$

$$\sum_{i=1}^{I} \underset{\sim}{H}^{B}_{a_i} = \underset{\sim}{H}_2 + \underset{\sim}{H}_4$$

$$\sum_{k=1}^{K} \underset{\sim}{H}^{B}_{c_k} = \underset{\sim}{H}_2 + \underset{\sim}{H}_6$$

$$\sum_{i=1}^{I} \underset{\sim}{H}^{C}_{a_i} = \underset{\sim}{H}_3 + \underset{\sim}{H}_5$$

$$\sum_{j=1}^{J} \underset{\sim}{H}^{C}_{b_j} = \underset{\sim}{H}_3 + \underset{\sim}{H}_6$$

In words, the sum of the simple effects hypothesis SSCP matrices for a factor across the levels of another factor equals the sum of the hypothesis SSCP matrix of the factor and the first order interaction hypothesis SSCP matrix over which the sum is being taken. The results established for the hypothesis SSCP matrices given in [52], [57] and [62] can serve as

computational checks and have been stated for the univariate case by Kirk (1968) and Winer (1971).

## Applying the STP to a Three-Way MANOVA

As overall hypothesis in terms of [13], one can choose $\underset{\sim}{C}_o'(IJK - 1 \times IJK) = [\underset{\sim}{I}, -\underset{\sim}{1}]$ where $\underset{\sim}{I}$ is an identity matrix of order IJK - 1, $\underset{\sim}{1}$ is a column matrix with IJK - 1 ones and $\underset{\sim}{A}_o = \underset{\sim}{I}_p$. This overall hypothesis implies the hypotheses in [45], [48], [53] and [58] as well as any hypothesis dealing with the joint nullity of a subset of contrasts among population means. Depending on the experiment and the number of tests in the family of interest to the experimenter, it may be possible to achieve greater power by using the Bonferroni inequality (Miller, 1966). Bird (1975) discusses the applicability of the Bonferroni inequality to hypotheses of the form [3] in which $\underset{\sim}{A}_i = \underset{\sim}{I}_p$ and indicates in what situations testing hypotheses with the Bonferroni procedure may be less conservative than the STP. In a completely unrestricted family of hypotheses associated with a MANOVA design, the only recourse is to use the STP.

## An Example in Two-Way MANOVA

To illustrate the application of the STP in analyzing a significant interaction, a small example using data from Timm (1975, page 409) will be used. The subclass means and within cells SSCP matrix are given in Table 1.

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Insert Table 1 about here

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The matrix $\underset{\sim}{\Xi}'(2 \times 6)$ in [21] is

$$[63] \qquad \underset{\sim}{\Xi}'(2 \times 6) = \begin{bmatrix} \xi^1_{11} & \xi^1_{12} & \xi^1_{21} & \xi^1_{22} & \xi^1_{31} & \xi^1_{32} \\ \xi^2_{11} & \xi^2_{12} & \xi^2_{21} & \xi^2_{22} & \xi^2_{31} & \xi^2_{32} \end{bmatrix}$$

As overall hypothesis in [13] choose $\underset{\sim}{A}_o = \underset{\sim}{I}_2$ and

$$[64] \qquad \underset{\sim}{C}_o'(5 \times 6) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Hypothesis test matrices to test [23], [25] and [27] can be formed by choosing

$$[65] \qquad \underset{\sim}{C}(2 \times 3) = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix}$$

and

$$\underset{\sim}{D}(1 \times 2) = [1 \quad -1]$$

respectively. Applying [24], [26] and [28] yields

$$[66] \qquad \underset{\sim}{C}_1'(2 \times 6) = \begin{bmatrix} 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\underset{\sim}{C}_2'(1 \times 6) = [1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1]$$

$$\underset{\sim}{C}_3'(2 \times 6) = \begin{bmatrix} 1 & -1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Hypothesis SSCP matrices for row, column and interaction effects and associated test statistics for the four criteria in [10] are given in Table 2 for both variables together and

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Insert Table 2 about here

---

individually. The critical regions for a .05 level test associated with a family of hypotheses implied by the overall hypothesis of equality of subclass mean vectors in our example are from [11]

[67] $$\psi_1^{.05} = \frac{\theta^{.05}(2, 1, 10.5)}{1 - \theta^{.05}(2, 1, 10.5)} = \frac{.455}{1 - .455} = .7953$$

$$\psi_2^{.05} = \upsilon_o^{.05}(2, 1, 10.5) = .9542$$

$$\psi_3^{.05} = \frac{1 - \upsilon^{.05}(2, 5, 24)}{\upsilon^{.05}(2, 5, 24)} = \frac{1 - .479323}{.479373} = 1.086$$

$$\psi_4^{.05} = \frac{V^{.05}(2, 1, 10.5)}{1 - V^{.05}(2, 1, 10.5)} = \frac{.589}{1 - .589} = 1.434$$

where from [9] $m_o = \frac{|2 - 5| - 1}{2} = 1$, $n_o = \frac{30 - 6 - 2 - 1}{2} = 10.5$ and $s_o = \min\{c_o, a_o\} = \min\{5, 2\} = 2$. In addition to the tables given in Timm (1975), the author found Grubb's (1954) tables reprinted in Roy, Gnanadesikan and Srivastava (1971) and Davis' (1970) tables useful for special cases of the exact distribution of the Hotelling-Lawley trace criterion. Critical statistics written in large numerals indicate significant results in Table 2 for the various criteria. Row, column and interaction effects

are significant on all criteria for both variables together. For the reading rate variable only significant row effects are obtained for three of the criteria. On reading comprehension significant column and interaction effects result with all four criteria. Since the interaction is significant, we proceed to simple row and column effects.

With $\underset{\sim}{C}$ from [65] substituted for $\underset{\sim}{G}_j$ in [32], the hypothesis test matrices associated with simple row effects of factor A at the first and second columns of B are

$$[68] \qquad \underset{\sim}{C}_4'(2 \times 6) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\underset{\sim}{C}_5'(2 \times 6) = \begin{bmatrix} 0 & 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

respectively.

With $\underset{\sim}{D}$ from [65] substituted for $\underset{\sim}{T}_i$ in [38], the hypothesis test matrices associated with simple column effects of factor B at the first, second and third levels of A are

$$[69] \qquad \underset{\sim}{C}_6'(1 \times 6) = [1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\underset{\sim}{C}_7'(1 \times 6) = [0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0]$$

$$\underset{\sim}{C}_8'(1 \times 6) = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1]$$

respectively. Hypothesis SSCP matrices and test statistics for the simple effects hypotheses are given in Table 3. Note

that for simple column effects only one eigenvalue is extracted from the product of an hypothesis SSCP and the inverse of the

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Insert Table 3 about here

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error SSCP. Thus all criteria reduce to the greatest root, $\Psi_1$. From Table 3 all four criteria indicate significant differences among the levels of factor A for the first level of factor B for both variables together and reading comprehension alone. Only $\Psi_1$ and $\Psi_2$ are significant for the reading rate variable. No differences were found among the levels of A for the second level of B. For simple column effects there are significant differences between the levels of B for the first and second levels of A, but not the third on both variables together and reading comprehension alone. As a computational check notice from Table 3 that equations [36] and [42] are satisfied, viz.,

$$[70] \quad \underset{\sim}{H}_4 + \underset{\sim}{H}_5 = \underset{\sim}{H}_1 + \underset{\sim}{H}_3$$

$$\underset{\sim}{H}_6 + \underset{\sim}{H}_7 + \underset{\sim}{H}_8 = \underset{\sim}{H}_2 + \underset{\sim}{H}_3$$

Having found differences among the levels of A for the first level of B, hypothesis test matrices for testing all pairwise comparisons among these levels are

[71] $\underset{\sim}{C}_9'\ (1 \times 6) = [1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0]$

$\underset{\sim}{C}_{10}'(1 \times 6) = [1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0]$

$\underset{\sim}{C}_{11}'(1 \times 6) = [0 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0]$

$\underset{\sim}{C}_9'$, $\underset{\sim}{C}_{10}'$ and $\underset{\sim}{C}_{11}'$ are associated with testing differences between $a_1$ and $a_2$ at $b_1$, $a_1$ and $a_3$ at $b_1$, and $a_2$ and $a_3$ at $b_1$, respectively. The hypothesis SSCP matrices and associated test statistics are given in Table 4.

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Insert Table 4 about here

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Note from Table 4 that significant differences between $a_1$ and $a_3$ at $b_1$ were found on both variables together using $\Psi_1$, but not individually. The other criteria do not yield significant results. Differences between $a_2$ and $a_3$ at $b_1$ are significant for all criteria on both variables together as well as singly. The family of eleven tests should be interpreted in conjunction with Figure 1 which has plots for the levels of each factor across the levels

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Insert Figure 1 about here

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of the other factor for both variables. Examining Tables 2, 3 and 4 along with Figure 1, the decisions associated with the STP can be summarized as follows using Roy's Largest root criterion. Considering both variables together significant row, column and interaction effects were found. Tests associated with simple effects indicated differences among the levels of A at $b_1$, but not at

$b_2$ and differences between the levels of B at $a_1$ and $a_2$ but not $a_3$. Further examination of pairwise differences within the levels of A at $b_1$ yielded differences between $a_1$ and $a_3$ at $b_1$, between $a_2$ and $a_3$ at $b_1$, but not $a_1$ and $a_3$ at $b_1$. Considering reading rate alone, only significant row effects, differences among the levels of A at $b_1$ and $a_2$ and $a_3$ at $b_1$ were found. For reading comprehension alone column and interaction effects were significant. Note, however, that differences among the levels of A at $b_1$, but not $b_2$ were found even though overall row effects were not present. Within the levels of A at $b_1$, only $a_2$ and $a_3$ differed at $b_1$. Significant differences were found between the levels of B at $a_1$ and $a_2$, but not $a_3$.

The calculations for the above simple example were carried out using MULTIVARIANCE (Finn, 1972). Although they will not be discussed in this paper, least square estimates and standard errors for all contrasts and discriminant functions for all hypotheses tested in the STP were obtained and could be used to aid interpretation. In this example we have illustrated how the STP allows the researcher to combine multivariate and univariate analysis in an attempt to isolate the sources of difference among population means when a significant interaction exists.

An Example in Three-Way MANOVA

Subclass mean vectors based on five observations per cell along with the within cells SSCP matrix for error are given in Table 5.

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Insert Table 5 about here

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The matrix $\underset{\sim}{\Xi}'(3 \times 8)$ in [44] is

[72]
$$\underset{\sim}{\Xi}'(3 \times 8) = \begin{bmatrix} \xi^1_{111} & \xi^1_{112} & \xi^1_{121} & \xi^1_{122} & \xi^1_{211} & \xi^1_{212} & \xi^1_{221} & \xi^1_{222} \\ \xi^2_{111} & \xi^2_{112} & \xi^2_{121} & \xi^2_{122} & \xi^2_{211} & \xi^2_{212} & \xi^2_{221} & \xi^2_{222} \\ \xi^3_{111} & \xi^3_{112} & \xi^3_{121} & \xi^3_{122} & \xi^3_{211} & \xi^3_{212} & \xi^3_{221} & \xi^3_{222} \end{bmatrix}$$

The hypothesis test matrix associated with the overall hypothesis of equality of population cell centroids is

[73] $\underset{\sim}{C}_o(7 \times 8) = [\underset{\sim}{I}(7 \times 7), -\underset{\sim}{1}(7 \times 1)]$

Hypothesis test matrices to test [45] can be formed by choosing

[74] $\underset{\sim}{C}(1 \times 2) = \underset{\sim}{D}(1 \times 2) = \underset{\sim}{E}(1 \times 2) = [1 \; -1]$

Applying [46] yields

[75] $\underset{\sim}{C}_1'(1 \times 8) = [1 \;\; 1 \;\; 1 \;\; 1 \;\; -1 \;\; -1 \;\; -1 \;\; -1]$

$\underset{\sim}{C}_2'(1 \times 8) = [1 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\; 1 \;\; -1 \;\; -1]$

$\underset{\sim}{C}_3'(1 \times 8) = [1 \;\; -1 \;\; 1 \;\; -1 \;\; 1 \;\; -1 \;\; 1 \;\; -1]$

$$\underset{\sim}{C}_4'(1 \times 8) = [1 \quad 1 \quad -1 \quad -1 \quad -1 \quad -1 \quad 1 \quad 1]$$

$$\underset{\sim}{C}_5'(1 \times 8) = [1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad 1]$$

$$\underset{\sim}{C}_6'(1 \times 8) = [1 \quad -1 \quad -1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 1]$$

$$\underset{\sim}{C}_7'(1 \times 8) = [1 \quad -1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1 \quad -1]$$

Table 6 contains the largest eigenvalues associated with testing the various hypotheses in [45] on all variables together as well as on subsets. Critical statistics written with large numerals are considered significant at the .05 level using Roy's Largest root criterion.

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Insert Table 6 about here

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The critical regions for a .05 level test associated with a family of hypotheses implied by [73] are from [11], $\Psi_1^{.05} = .8692$, $\Psi_2^{.05} = 1.2678$, $\Psi_3^{.05} = 1.6827$ and $\Psi_4^{.05} = 3.926$, where $s_o = \min\{7, 3\} = 3$, $m_o = \frac{3}{2}$ and $n_o = 14$. Since all the hypotheses in this example involve one root, Roy's Largest root will be used, since it is more parsimonious and possibly more resolvent than the other criteria. From Table 6 all hypotheses are rejected for all variables analyzed together except $H_3^C$. Since the three factor interaction is significant, we proceed to simple interaction effects.

The hypothesis test matrices in [49] for the hypotheses in [48] can be written

[76] $H^{AB}_{c_1}$: $\underset{\sim}{C}'_{c_1}$ = [1 0 -1 0 -1 0 1 0]

$H^{AB}_{c_2}$: $\underset{\sim}{C}'_{c_2}$ = [0 1 0 -1 0 -1 0 1]

$H^{AC}_{b_1}$: $\underset{\sim}{C}'_{b_1}$ = [1 -1 0 0 -1 1 0 0]

$H^{AC}_{b_2}$: $\underset{\sim}{C}'_{b_2}$ = [0 0 1 -1 0 0 -1 1]

$H^{BC}_{a_1}$: $\underset{\sim}{C}'_{a_1}$ = [1 -1 -1 1 0 0 0 0]

$H^{BC}_{a_2}$: $\underset{\sim}{C}'_{a_2}$ = [0 0 0 0 1 -1 -1 1]

Table 7 contains the largest eigenvalues with those significant written in large numerals. Five of the six simple interactions

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Insert Table 7 about here

---

are significant for all three variables together. To aid in the analysis of these simple interactions, Table 8 contains the largest roots for all the simple simple effects hypotheses.

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Insert Table 8 about here

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The hypothesis test matrices in [54] for the simple simple effects hypotheses in [53] for our data are

[77] $H^{A}_{bc_{11}}$: $\underset{\sim}{C}'_{11}$ = [1 0 0 0 -1 0 0 0]

$H^{A}_{bc_{12}}$: $\underset{\sim}{C}'_{12}$ = [0 1 0 0 0 -1 0 0]

$H^{A}_{bc_{21}}$: $\underset{\sim}{C}'_{21}$ = [0 0 1 0 0 0 -1 0]

$H^{A}_{bc_{22}}$: $\underset{\sim}{C}'_{22}$ = [0 0 0 1 0 0 0 -1]

$H^{B}_{ac_{11}}$: $\underset{\sim}{C}'_{11}$ = [1 0 -1 0 0 0 0 0]

$H^{B}_{ac_{12}}$: $\underset{\sim}{C}'_{12}$ = [0 1 0 -1 0 0 0 0]

$H^{B}_{ac_{21}}$: $\underset{\sim}{C}'_{21}$ = [0 0 0 0 1 0 -1 0]

$H^{B}_{ac_{22}}$: $\underset{\sim}{C}'_{22}$ = [0 0 0 0 0 1 0 -1]

$H^{C}_{ab_{11}}$: $\underset{\sim}{C}'_{11}$ = [1 -1 0 0 0 0 0 0]

$H^{C}_{ab_{12}}$: $\underset{\sim}{C}'_{12}$ = [0 0 1 -1 0 0 0 0]

$H^{C}_{ab_{21}}$: $\underset{\sim}{C}'_{21}$ = [0 0 0 0 1 -1 0 0]

$H^{C}_{ab_{22}}$: $\underset{\sim}{C}'_{22}$ = [0 0 0 0 0 0 1 -1]

We shall consider interpreting the simple AB interaction with the first level of factor C fixed. Interpretation of the others is analogous. Figure 2 contains graphs of these simple interactions. Examining Figure 2 in conjunction with

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Insert Figure 2 about here

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Tables 7 and 8, we can say that for the first level of factor C being fixed there are significant differences between the levels of A at $b_1$ for all three variables together, on subsets of two alone and on variable one singly. However, no differences

between the levels of A were found on any variables at $b_2$ for $c_1$ fixed. For $c_1$ fixed, significant differences between the levels of B at $a_2$ for all three variables together, on subsets of two alone and on variable one singly were found. No differences exist between the levels of B at $a_1$ for $c_1$ fixed. It is interesting to note from Table 6 that no differences between the levels of C were found when averaged across the levels of the other factors, but significant differences were obtained for all variables together for all possible pairs of fixed levels of factors A and B. Undoubtedly, the interpretation of significant main effects must be done considering significant interactions as this example indicates.

The calculations for this data were carried out using MULTIVARIANCE (Finn, 1972). Computational checks using [52] for simple interactions and [57] for simple simple effects were verified. MULTIVARIANCE (Finn, 1972) gives least squares estimates and standard errors for all contrasts. Discriminant function analysis and step-down statistics for each hypothesis are given and can be used with the STP to aid interpretation. Hypothesis mean products for each hypothesis can be obtained. From these, hypothesis SSCP matrices are derived and used for computational checks.

## Summary

The formulation of hypotheses and associated test statistics in the context of Gabriel's (1968) simultaneous test procedure (STP) for the unrestricted full rank general linear

multivariate model were presented. Using properties of Kronecker products of matrices, it was shown how to formulate overall hypotheses and hypotheses associated with analyzing significant interactions in two-way and three-way factorial designs. Extension to higher order factorials should present no difficulty. Examples were given to illustrate how to use this approach to analyze and interpret main effects in the light of significant interactions. All calculations were carried out using MULTIVARIANCE (Finn, 1972).

## References

Bird, K. D. Simultaneous contrast testing procedures for multivariate experiments. Multivariate Behavioral Research, 1975, 10, 343-351.

Bock, R. D. Multivariate statistical methods in behavioral research. New York: McGraw-Hill, 1975.

Davis, A. W. Exact distributions of Hotelling's generalized $T_0^2$. Biometrika, 1970, 57, 187-191.

Finn, J. D. MULTIVARIANCE: Univariate and multivariate analysis of variance, covariance and regression (Version V). Chicago: International Educational Services, 1972.

Finn, J. D. A general model for multivariate analysis. New York: Holt, Rinehart and Winston, Inc., 1974.

Gabriel, K. R. Simultaneous test procedures in multivariate analysis of variance. Biometrika, 1968, 55, 489-504.

Gabriel, K. R. A comparison of some methods of simultaneous inference in MANOVA. In P. R. Krishnaiah (Ed.), Multivariate analysis--II. New York: Academic Press, 1969a.

Gabriel, K. R. Simultaneous test procedures--some theory of multiple comparisons. The Annals of Mathematical Statistics, 1969b, 40(1), 224-250.

Gabriel, K. R. A procedure for testing the homogeneity of all sets of means in analysis of variance. Biometrics, 1964, 20, 459-477.

Kirk, R. E. Experimental design: Procedures for the behavioral sciences. Monterey, Calif.: Brooks/Cole, 1968.

Miller, R. G. Simultaneous statistical inference. New York: McGraw-Hill, 1966.

Olson, C. L. A Monte Carlo investigation of the robustness of multivariate analysis of variance. Unpublished doctoral dissertation, University of Toronto, 1973.

Olson, C. L. Comparative robustness of six tests in multivariate analysis of variance. Journal of the American Statistical Association, 1974, 69, 894-908.

Roy, S. N., Gnanadesikan, R., & Srivastava, J. N. Analysis and design of certain quantitative multi-response experiments. New York: Pergamon Press, 1971.

Timm, N. H. Multivariate analysis with applications in education and psychology. Monterey, Calif.: Brooks/Cole, 1975.

Timm, N. H., & Carlson, J. E. Analysis of variance through full rank models. Multivariate Behavioral Research Monograph, 1975, No. 75-1.

Winer, B. J. Statistical principles in experimental design (2nd ed.). New York: McGraw-Hill, 1971.

Table 1

Cell Means and Error SSCP for Illustrative Data

| | $b_1$ | | $b_2$ | |
|---|---|---|---|---|
| | RR | RC | RR | RC |
| $a_1$ | 11.00 | 21.20 | 9.20 | 15.80 |
| $a_2$ | 13.40 | 24.80 | 10.20 | 16.80 |
| $a_3$ | 8.00 | 17.20 | 8.80 | 19.00 |

NOTE. RR = Reading Rate
RC = Reading Comprehension

$$\underset{\sim}{E} = \begin{bmatrix} 45.60 & \\ 19.80 & 56.00 \end{bmatrix}$$

Table 2

Hypothesis SSCP Matrices for Row, Column and Interaction Effects and Associated Test Statistics

| Hypothesis | SSCP | Test Statistic | RR,RC | RR | RC |
|---|---|---|---|---|---|
| $H_1$ | $\begin{bmatrix} 57.8 & \\ 45.9 & 42.466 \end{bmatrix}$ | $\Psi_1$ | 1.441 | 1.268 | .758 |
| | | $\Psi_2$ | 1.553 | 1.268 | .758 |
| | | $\Psi_3$ | 1.712 | 1.268 | .758 |
| | | $\Psi_4$ | 2.237 | 1.268 | .758 |
| $H_2$ | $\begin{bmatrix} 14.7 & \\ 40.6 & 112.133 \end{bmatrix}$ | $\Psi_1$ | 2.003 | .322 | 2.002 |
| | | $\Psi_2$ | 2.003 | .322 | 2.002 |
| | | $\Psi_3$ | 2.003 | .322 | 2.002 |
| | | $\Psi_4$ | 2.003 | .322 | 2.002 |
| $H_3$ | $\begin{bmatrix} 20.6 & \\ 51.3 & 129.867 \end{bmatrix}$ | $\Psi_1$ | 2.307 | .452 | 2.319 |
| | | $\Psi_2$ | 2.312 | .452 | 2.319 |
| | | $\Psi_3$ | 3.329 | .452 | 2.319 |
| | | $\Psi_4$ | 2.362 | .452 | 2.319 |

Table 3

Hypothesis SSCP Matrices for Simple Effects of Factor A and Factor B with Associated Test Statistics

| Hypothesis | SSCP | Test Statistic | RR,RC | RR | RC |
|---|---|---|---|---|---|
| $H_4$ | $\begin{bmatrix} 73.2 & \\ 102.8 & 144.533 \end{bmatrix}$ | $\Psi_1$ | 3.060 | 1.053 | 2.581 |
| | | $\Psi_2$ | 3.062 | 1.053 | 2.581 |
| | | $\Psi_3$ | 3.068 | 1.053 | 2.581 |
| | | $\Psi_4$ | 2.090 | 1.053 | 2.581 |
| $H_5$ | $\begin{bmatrix} 5.2 & \\ -5.6 & 26.8 \end{bmatrix}$ | $\Psi_1$ | .735 | .114 | .479 |
| | | $\Psi_2$ | .803 | .114 | .479 |
| | | $\Psi_3$ | .853 | .114 | .479 |
| | | $\Psi_4$ | .950 | .114 | .479 |
| $H_6$ | $\begin{bmatrix} 8.1 & \\ 24.3 & 72.9 \end{bmatrix}$ | $\Psi_1$ | 1.303 | .178 | 1.302 |
| $H_7$ | $\begin{bmatrix} 25.6 & \\ 64.0 & 160.0 \end{bmatrix}$ | $\Psi_1$ | 2.866 | .561 | 2.857 |
| $H_8$ | $\begin{bmatrix} 1.6 & \\ 3.6 & 8.1 \end{bmatrix}$ | $\Psi_1$ | .146 | .035 | .148 |

Table 4

Hypothesis SSCP Matrices and Test Statistics for Pairwise Comparisons Among the Levels of A at the First Level of B

| Hypotheses | SSCP | Test Statistic | RR,RC | RR | RC |
|---|---|---|---|---|---|
| $H_9$ | $\begin{bmatrix} 14.4 & \\ 21.6 & 32.4 \end{bmatrix}$ | $\Psi_1$ | .6609 | .316 | .579 |
| $H_{10}$ | $\begin{bmatrix} 22.5 & \\ 30.0 & 40.0 \end{bmatrix}$ | $\Psi_1$ | 8.771 | .493 | .714 |
| $H_{11}$ | $\begin{bmatrix} 72.9 & \\ 102.6 & 144.4 \end{bmatrix}$ | $\Psi_1$ | 3.055 | 1.598 | 2.58 |

Table 5

Subclass Mean Vectors and Within Cells

SSCP for Three-Way MANOVA Data

| | | $c_1$ | | | $c_2$ | | |
|---|---|---|---|---|---|---|---|
| | | Var 1 | Var 2 | Var 3 | Var 1 | Var 2 | Var 3 |
| $a_1$ | $b_1$ | 10.0 | 14.0 | 12.0 | 14.0 | 17.8 | 9.0 |
| | $b_2$ | 15.0 | 25.0 | 18.0 | 16.0 | 17.0 | 16.0 |
| $a_2$ | $b_1$ | 24.0 | 8.0 | 16.0 | 8.0 | 14.0 | 13.0 |
| | $b_2$ | 8.0 | 13.0 | 10.0 | 16.0 | 17.0 | 12.0 |

$$\underset{\sim}{E} = \begin{bmatrix} 276.0 & & \\ 386.0 & 660.8 & \\ 217.0 & 335.0 & 254 \end{bmatrix}$$

Table 6

Test Statistics for Hypotheses in [45]

| Hypothesis | Variables | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1, 2, 3 | 1, 2 | 1, 3 | 2, 3 | 1 | 2 | 3 |
| $H_1^A$ | 2.7962 | 2.7830 | .1740 | .8187 | .0023 | .4495 | .0394 |
| $H_2^B$ | 2.010 | 1.9869 | .3474 | .3906 | .0023 | .3133 | .0886 |
| $H_3^C$ | .8609 | .5366 | .1198 | .6253 | .0204 | .0318 | .0886 |
| $H_4^{AB}$ | 3.7580 | 2.3315 | 1.0143 | 2.6528 | .5095 | .0046 | .9842 |
| $H_5^{AC}$ | 6.0787 | 5.8005 | 1.8985 | .2667 | .3827 | .1907 | .0394 |
| $H_6^{BC}$ | 10.6325 | 10.6280 | 1.8267 | 1.4343 | .9986 | .1801 | .0886 |
| $H_7^{ABC}$ | 6.5124 | 5.6904 | 3.8755 | .0978 | 1.6508 | .0908 | .0394 |

Table 7

Test Statistics for Hypotheses in [48]

| Hypothesis | Variables | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1, 2, 3 | 1, 2 | 1, 3 | 2, 3 | 1 | 2 | 3 |
| $H^{AB}_{c_1}$ | 7.6825 | 7.6411 | 2.3023 | 1.2599 | 1.9973 | .0681 | .7087 |
| $H^{AB}_{c_2}$ | 2.5880 | .3809 | 2.5874 | 1.4907 | .16304 | .02732 | .31496 |
| $H^{AC}_{b_1}$ | 12.2468 | 11.2186 | 5.5181 | .0276 | 1.8116 | .00916 | .0000 |
| $H^{AC}_{b_2}$ | .3443 | .2724 | .2558 | .3367 | .2219 | .2724 | .07874 |
| $H^{BC}_{a_1}$ | 1.1125 | .6383 | .2099 | .9874 | .04976 | .26339 | .00492 |
| $H^{BC}_{a_2}$ | 16.0324 | 15.6801 | 5.4923 | .5447 | 2.6087 | .00757 | .12303 |

Table 8

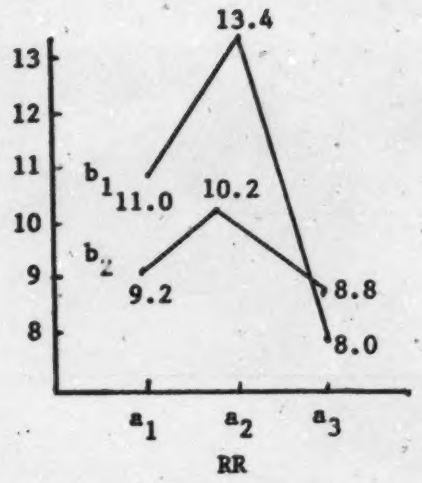
Test Statistics for Hypotheses in [53]

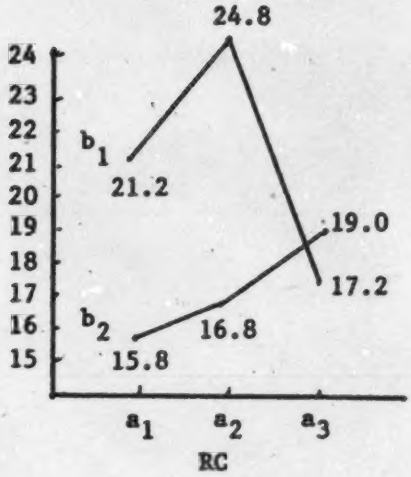
| Hypothesis | Variables | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1, 2, 3 | 1, 2 | 1, 3 | 2, 3 | 1 | 2 | 3 |
| $H^{A}_{bc_{11}}$ | 15.3028 | 15.2987 | 3.2474 | 1.6090 | 1.7754 | .1362 | .1575 |
| $H^{A}_{bc_{12}}$ | 2.6429 | .7618 | 2.6044 | 1.0979 | .3261 | .0546 | .1575 |
| $H^{A}_{bc_{21}}$ | .6675 | .5448 | .6307 | .6539 | .4438 | .5448 | .6299 |
| $H^{A}_{bc_{22}}$ | .5321 | .0000 | .4797 | .4752 | .0000 | .0000 | .1585 |
| $H^{B}_{ac_{11}}$ | .5933 | .5583 | .3548 | .4631 | .2264 | .4578 | .3543 |
| $H^{B}_{ac_{12}}$ | 1.6508 | .3037 | .9194 | 1.6314 | .0362 | .0024 | .4823 |
| $H^{B}_{ac_{21}}$ | 17.8195 | 17.8093 | 3.6168 | 2.2582 | 2.3188 | .0946 | .3543 |
| $H^{B}_{ac_{22}}$ | 2.8497 | 1.9655 | 2.1729 | .2228 | .57971 | .0341 | .0098 |
| $H^{C}_{ab_{11}}$ | 1.2834 | .2115 | 1.2770 | .7755 | .1449 | .0546 | .0886 |
| $H^{C}_{ab_{12}}$ | 1.8385 | 1.8347 | .2418 | .3676 | .0091 | .2421 | .0394 |
| $H^{C}_{ab_{21}}$ | 19.1206 | 18.9617 | 5.0702 | 1.2204 | 2.3188 | .1362 | .0886 |
| $H^{C}_{ab_{22}}$ | 1.8420 | 1.6477 | 1.1314 | .0606 | .5797 | .0605 | .0394 |

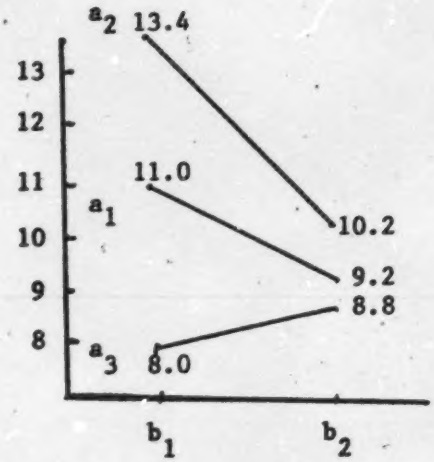
## Figure Captions

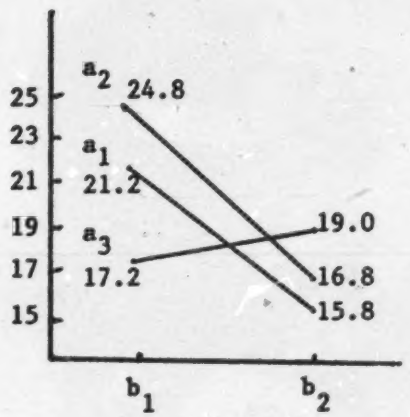
Figure 1. Plots of the cell means for reading rate (RR) and reading comprehension (RC).

Figure 2. Plots of cell means for simple AB interaction with $c_1$ fixed.

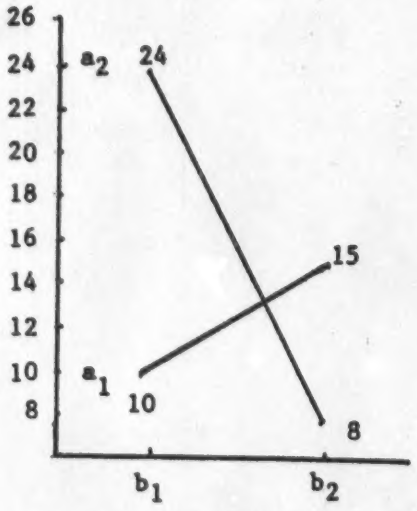

13.4
13
12
11
10
9
8
$b_1$
11.0
10.2
$b_2$
9.2
8.8
8.0
$a_1$
$a_2$
$a_3$
RR


24.8
24
23
22
21
20
19
18
17
16
15
$b_1$
21.2
19.0
17.2
$b_2$
16.8
15.8
$a_1$
$a_2$
$a_3$
RC

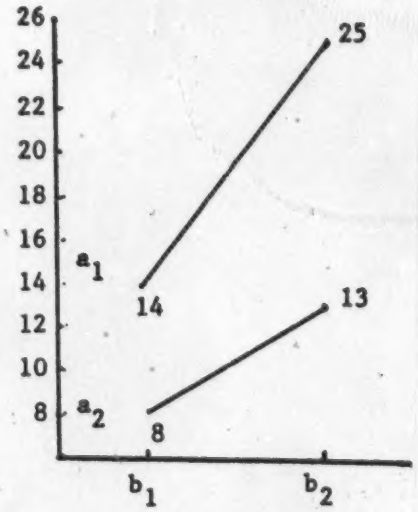

$a_2$ 13.4
13
12
11
10
9
8
11.0
$a_1$
10.2
9.2
8.8
$a_3$
8.0
$b_1$
$b_2$
RR


$a_2$ 24.8
25
23
$a_1$
21
21.2
19
$a_3$
19.0
17
17.2
16.8
15
15.8
$b_1$
$b_2$
RC

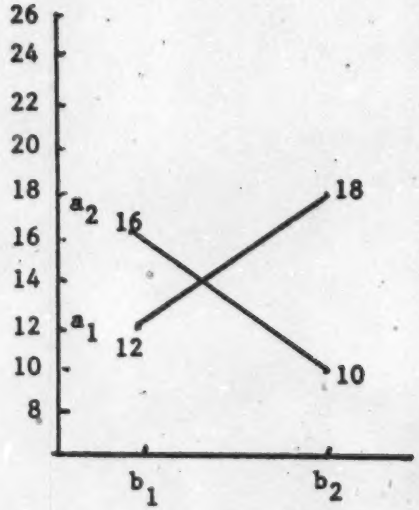
Variable 1



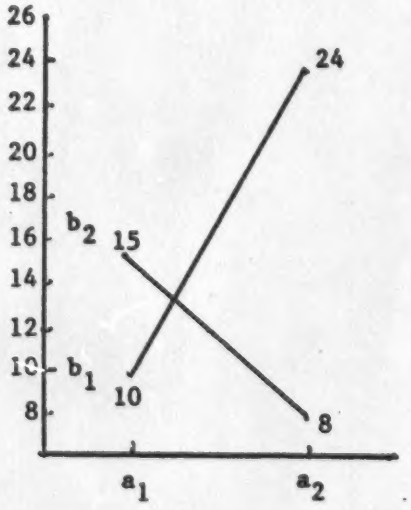

Variable 2



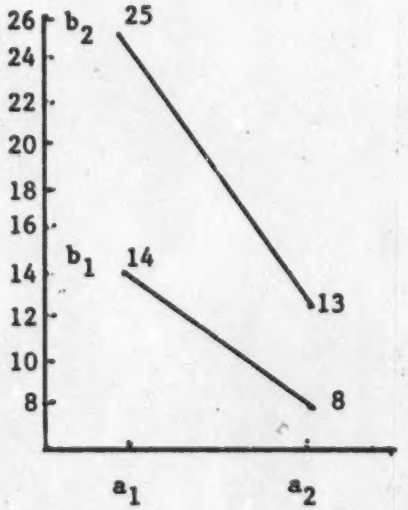

Variable 3




Variable 1




Variable 2




Variable 3